THE COHOMOLOGY ALGEBRA OF UNORDERED CONFIGURATION SPACES

YVES FÉLIX AND DANIEL TANRÉ

ABSTRACT. Given an N-dimensional compact manifold M and a field $\mathbb{k}$, F. Cohen and L. Taylor have constructed a spectral sequence, $\mathcal{E}(M,n,\mathbb{k})$, converging to the cohomology of the space of ordered configurations of n points in M. The symmetric group Σ_n acts on this spectral sequence giving a spectral sequence of Σ_n differential graded commutative algebras. Here, we provide an explicit description of the invariants algebra $(E_1,d_1)^{\Sigma_n}$ of the first term of $\mathcal{E}(M,n,\mathbb{Q})$. We apply this determination in two directions:

– in the case of a complex projective manifold or of an odd dimensional manifold M, we obtain the cohomology algebra $H^*(C_n(M);\mathbb{Q})$ of the space of unordered configurations of n points in M (the concrete example of $P^2(\mathbb{C})$ is detailed),

– we prove the degeneration of the spectral sequence formed of the Σ_n-invariants $\mathcal{E}(M,n,\mathbb{Q})^{\Sigma_n}$ at level 2, for any manifold M.

These results use a transfer map and are also true with coefficients in a finite field $\mathbb{F}_p$ with $p>n$.

Given an N-dimensional compact manifold, M, define the space $F(M,n)$ of *ordered configurations* of n points in M, as

$$F(M,n)=\{\,(x_1,x_2,\dots,x_n)\,|\,x_i\neq x_j\,,\text{for } i\neq j\,\}\,.$$

The symmetric group Σ_n acts freely on $F(M,n)$ by permutation of coordinates. Here we are interested in the orbit space $C_n(M)=F(M,n)/\Sigma_n$ called the space of *unordered configurations* of n points in M. The determination of the homotopy type, even of the rational homotopy type, of the spaces $F(M,n)$ and $C_n(M)$ is a hard problem.

Let $\mathbb{k}$ be a field. The principal tool for the determination of the cohomology algebra of $F(M,n)$ is a spectral sequence $\mathcal{E}(M,n,\mathbb{k})$ of graded commutative algebras constructed by F. Cohen and L. Taylor in [7]. This spectral sequence converges to $H^*(F(M,n);\mathbb{k})$ and has a E_1-term that we briefly recall now:

$$(E_1,d_1)=((H^*(M;\mathbb{k})^{\otimes n}\otimes\wedge_{1\leq i,j\leq n}e_{ij})/I,d_1)\,.$$

The elements of $(\otimes^n H(M;\mathbb{k}))^t$ have bidegree $(t,0)$, the e_{ij} bidegree $(0,N-1)$ and the differential denoted by d_1 is the first non-zero differential. The symbol $\wedge V$ denotes the free commutative graded algebra on the graded vector space V. The ideal I is generated by the elements $e_{ij}e_{jk}+e_{jk}e_{ki}+e_{ki}e_{ij}$, e_{ij}^2, $e_{ij}-(-1)^N e_{ji}$ and $(p_i^*(a)-p_j^*(a))\otimes e_{ij}$ where $p_i:M^n\to M$ is the projection on the i^{th} factor. Finally,

$$d_1(e_{ij})=p_{ij}^*(\Delta),$$

where $p_{ij}:M^n\to M^2$ is the projection on the i^{th} and the j^{th} factors and $\Delta\in H^N(M^2;\mathbb{k})$ is the diagonal class.

The symmetric group Σ_n acts on (E_1,d_1) by permuting the coordinates in $H^*(M;\mathbb{k})^{\otimes n}$ and on $\wedge_{1\leq i,j\leq n}e_{ij}$ by $\sigma(e_{ij})=e_{\sigma(i)\sigma(j)}$. With this action, the spectral sequence $\mathcal{E}(M,n,\mathbb{k})$

becomes a spectral sequence of Σ_n-algebras and creates a spectral sequence $\mathcal{E}(M,n,\Bbbk)^{\Sigma_n}$ consisting of the Σ_n-invariants of each stage. With the basic results recalled in Section 1, observe that, for $\Bbbk = \mathbb{Q}$, or for $\Bbbk = \mathbb{F}_p$ and $p > n$, the spectral sequence $\mathcal{E}(M,n,\Bbbk)^{\Sigma_n}$ converges to $H^*(C_n(M);\Bbbk)$.

This work is concerned with the study of the degeneration of the spectral sequence $\mathcal{E}(M,n,\Bbbk)^{\Sigma_n}$ and its implication in the determination of the cohomology algebra of $H^*(C_n(M);\Bbbk)$.

First recall the particular situation of a smooth projective complex manifold M and $\Bbbk = \mathbb{Q}$. I. Kriz ([14]) and B. Totaro ([19]) prove that the differential graded algebra (E_1, d_1) is quasi-isomorphic to the Sullivan minimal model of the space $F(M,n)$ which means that (E_1, d_1) contains all the rational homotopy type of $F(M,n)$. In particular, in this case, the spectral sequences $\mathcal{E}(M,n,\mathbb{Q})$ and $\mathcal{E}(M,n,\mathbb{Q})^{\Sigma_n}$ collapse at level 2 and $H^*(C_n(M);\mathbb{Q}) \cong H(E_1,d_1)^{\Sigma_n}$ as algebra (see [9, Theorem 8 and Corollary 8c], [14, Remark 1.3] and [19, Corollary 1 of Theorem 5]). For more general manifold we do not have a model for the rational homotopy type except for $n = 2$ by a work of P. Lambrechts and D. Stanley ([15]).

As we see in the particular case of a projective complex manifold and $\Bbbk = \mathbb{Q}$, we have first to understand the invariants algebra $(E_1, d_1)^{\Sigma_n}$. We do that for any field $\Bbbk$ and prove in Theorem 6 of Section 3 that, for N even, we have an isomorphism

$$E_1^{\Sigma_n} \cong \Gamma_n^H \,, \text{ with } \Gamma_n^H = \oplus_{r=0}^{[n/2]} \Gamma^{n-2r}(H) \otimes \Gamma^r(s^{N-1}H),$$

where $\Gamma^j(V) \subset T^j(V)$ denotes the invariants subspace for the action of the symmetric group Σ_j by permutation of the factors. The law structure induced on Γ_n^H by this isomorphism is described in Theorem 7. In the case $\Bbbk = \mathbb{Q}$ or $\Bbbk = \mathbb{F}_p$ with $p > n$, this result becomes

Theorem 1. *Let M be a N-dimensional compact manifold with N even and let $\Bbbk = \mathbb{Q}$ or $\Bbbk = \mathbb{F}_p$ with $p > n$. We denote by $\bullet$ the multiplication law of the cohomology algebra $H = H^*(M;\mathbb{Q})$. Then there is an isomorphism of graded algebras*

$$(E_1, d_1)^{\Sigma_n} \cong (C_n^H, d) \,, \textit{ with } C_n^H = \left(\oplus_{r=0}^{[n/2]} \wedge^{n-2r}(H) \otimes \wedge^r (s^{N-1}H), d \right).$$

where the multiplication law on C_n^H depends only on the multiplication law $\bullet$ on H and is described in Theorem 8. The differential

$$d\colon \wedge^{n-2r}(H) \otimes \wedge^r(s^{N-1}H) \to \wedge^{n-2r+2}(H) \otimes \wedge^{r-1}(s^{N-1}H)$$

is determined by

$$d(e_0 \wedge \ldots \wedge e_0 \otimes s^{N-1}a) = \frac{1}{2} \sum_k (-1)^{|b'_k|} e_0 \wedge \ldots \wedge e_0 \wedge (a \bullet b_k) \wedge b'_k \,,$$

where e_0 is the unit of H, the $\{b_i\}$ form an homogeneous basis for H and the $\{b'_i\}$ are the dual basis for Poincaré duality.

The differential d of a general element is expressed in Definition 8. If we consider $C^H = \oplus_n C_n^H = \wedge H \otimes \wedge(s^{N-1}H)$, it appears from Definition 8 that the differential d is also a differential of algebra for the *usual product* on $\wedge H \otimes \wedge(s^{N-1}H)$. Denote by $\Gamma_{sec}(M)$ the space of sections of the sphere bundle associated to the tangent bundle of M. The commutative differential graded algebra $\wedge H \otimes \wedge(s^{N-1}H)$ with the usual product is the model of $\Gamma_{sec}(M)$ already determined in [11]. The fact that our C_n^H fit together

and give this model corresponds to the link between $\Gamma_{sec}(M)$ and $\vee_{n\geq 0}C_n(M)$ revealed in [16], see also [3].

In the case of a complex projective manifold M, the Kriz-Totaro result recalled above gives an isomorphism of algebras between $H^*(C_n(M);\mathbb{Q})$ and $H(C_n^H,d)$. This description is effective and in Section 5, we can determine explicitely the rational cohomology algebra of the unordered configuration spaces of n points in $P^2(\mathbb{C})$:

Theorem 2. *The rational cohomology of the unordered configuration spaces of n points into $P^2(\mathbb{C})$ is given by*

$$\begin{array}{lll} n=1,2 & \wedge x/x^3 & |x|=2\,; \\ n=3 & \wedge(x,y)/(x^3,yx^2) & |x|=2\,,|y|=3\,; \\ n\geq 4 & \wedge(x,y)/x^3 & |x|=2\,,|y|=3\,. \end{array}$$

In the more general case of a real manifold, recall that the homology vector space $H_*(C_n(M);\Bbbk)$ has been described, when N is odd, by C.-F. Bödigheimer, R. Cohen and L. Taylor in [4], and, when N is even and $\Bbbk=\mathbb{Q}$, by Y. Félix and J.-C. Thomas in [11]. From these results, we deduce:

Theorem 3. *For any manifold M, the spectral sequence formed of the Σ_n-invariants, $\mathcal{E}(M,n,\mathbb{Q})^{\Sigma_n}$, collapses at level 2.*

Observe that this theorem contrasts with a recent result of the first author and J.-C. Thomas. Answering a question of M. Bendersky and S. Gitler ([2]), they prove that there exist manifolds M for which the spectral sequence $\mathcal{E}(M,n,\mathbb{Q})$ does not collapse at level 2, see [12].

The result of [4] being true over the field $\mathbb{F}_p$, we have also

Theorem 4. *Let M be an odd dimensional manifold and let $\Bbbk=\mathbb{Q}$ or $\Bbbk=\mathbb{F}_p$ with $p>n$. The spectral sequence $\mathcal{E}(M,n,\Bbbk)^{\Sigma_n}$ collapses at level 2 and the cohomology algebra of the unordered configuration space is isomorphic to $\wedge^n(H)$.*

As a conclusion, observe that two cases are not covered by our study and they lead to

Open Problem 1. *Does the spectral sequence $\mathcal{E}(M,n,\mathbb{F}_p)^{\Sigma_n}$ collapse for an even dimensional manifold?*

What is missing for answering is the knowledge of the homology $H_*(C_n(M);\mathbb{F}_p)$ as we have it over $\mathbb{Q}$ in the even dimensional case. Notice now that Theorem 3 gives an information on the algebra structure of $H^*(C_n(M);\mathbb{Q})$ only up to a filtration.

Open Problem 2. *Can we recover the algebra structure of $H^*(C_n(M);\mathbb{Q})$ in all cases?*

We know the algebra structure when M is odd dimensional, when M is projective complex or when $n=2$ (see [15]) but not in the general case.

In Section 1, we recall some basic facts concerning equivariant homology and cohomology. Section 2 and Section 3 are concerned with the space of Σ_n-invariants of $H^{(n-1)(N-1)}(F(\mathbb{R}^N,n);\Bbbk)$ and E_1 respectively. Section 4 is devoted to the differential graded algebra (C_n^H,d) and the proof of Theorem 1. It contains also the proof of Theorem 3 in the even dimensional case. As an example we give in Section 5 a complete description of the cohomology algebra of the configuration spaces of n points in the

complex projective space $P^2(\mathbb{C})$. In Section 6, we study the case of odd dimensional manifolds and prove Theorem 4.

1. Equivariant homology

Let G be a finite group and let $\Bbbk$ be a field. A *G-cdga* is a commutative differential graded algebra (A, d_A) with $H^0(A, d_A) = \Bbbk$, on which G acts by cdga maps. The invariant subspace $(A, d_A)^G$ defines a sub cdga of (A, d_A). Recall some basic facts on these objects:

Proposition 1. *Let $\Bbbk$ be a field of characteristic that does not divide the order of G and let $f : (A, d_A) \to (B, d_B)$ be a G-equivariant quasi-isomorphism. Then, we have*
1) $f^G : (A, d_A)^G \to (B, d_B)^G$ is also a quasi-isomorphism;
2) $H\left((A, d_A)^G\right) = (H(A, d_A))^G$.

Proof. 1) Let $a \in (A, d_A)^G$ be a cocycle and suppose that $f^G(a) = d(b)$, then $f(a) = db$ and $a = d(c)$. Since a is invariant, $a = d\left(\frac{1}{|G|}\sum_{g\in G} g\cdot c\right)$. This shows that $H(f^G)$ is injective.

Let now $b \in B^G$ be a cocycle. Since f is a quasi-isomorphism, there is a cocycle $a \in A$ and an element $c \in B$ such that $f(a) = b + d(c)$. Therefore $f\left(\frac{1}{|G|}\sum_{g\in G} g\cdot a\right) = b + d\left(\frac{1}{|G|}\sum_{g\in G} g\cdot c\right)$. Thus $H(f^G)$ is also surjective.

2) The second statement is a consequence of the existence of the canonical inclusion $A^G \to A$ and of $\sigma\colon A \to A^G$, $\sigma(a) = \sum_{g\in G} g\cdot a$. □

Let X be a simplicial complex with a (simplicial) action of G. Recall from G. Bredon ([5, Page 115]) that the action is *regular* if, for any $g_0, \dots, g_n$ in G and simplices $(v_0, \dots, v_n)$, $(g_0v_0, \dots, g_nv_n)$ of X, there exists an element $g \in G$ such that $gv_i = g_iv_i$ for all i. By [5, Proposition 1.1, Page 116], the induced action on the second barycentric subdivision is always regular. Here, *by a G-space, we mean a connected simplicial complex on which G acts regularly.*

Denote by $C(X)$ the oriented chain complex of X and observe that $C(X)$ is a module over the group ring $\mathbb{Z}[G]$ of G. The canonical simplicial map $\rho\colon X \to X/G$ induces $\rho_*\colon C(X) \to C(X/G)$. Define now $\sigma\colon C(X) \to C(X)$, $c \mapsto \sum_{g\in G} gc$. One has $\mathrm{K}er\,\sigma = \mathrm{K}er\,\rho_*$. Therefore σ induces $\overline{\sigma}$

$$\begin{tikzcd} C(X) \arrow[r, "\sigma"] \arrow[d, "\rho_*"'] & C(X) \\ C(X/G) \arrow[ur, "\overline{\sigma}"'] & \end{tikzcd}$$

such that $\overline{\sigma}\circ\rho_* = \sigma$. One can prove:

Proposition 2 ([5, Page 120]). *Let $\Bbbk$ be a field of characteristic that does not divide the order of G and let X be a G-space. Then there are isomorphisms*

$$C_*(X/G; \Bbbk) \cong C_*(X; \Bbbk)^G \text{ and } C^*(X/G; \Bbbk) \cong C^*(X; \Bbbk)^G\,.$$

In the case $\Bbbk = \mathbb{Q}$, when X is a G-space the group G acts on the Sullivan algebra of PL-forms on X, $A_{PL}(X)$, ([18]). A similar argument identifies $A_{PL}(X/G)$ with $A_{PL}(X)^G$. Moreover, a G-cdga (A, d_A) admits a minimal model

$$\varphi : (\wedge V, d) \xrightarrow{\simeq} (A, d_A)$$

with an action of G on $(\wedge V, d)$ making φ G-equivariant ([13]). This model is unique up to G-isomorphisms. We call it *a G-minimal model* and Proposition 1 implies

Proposition 3. *Let $(\wedge V, d) \stackrel{\cong}{\to} A_{PL}(X)$ be a G-minimal model of the G-space X, then the fixed point set $(\wedge V, d)^G$ is a model for $A_{PL}(X/G)$.*

This means that the cdga's $(\wedge V, d)^G$ and $A_{PL}(X/G)$ have the same minimal model.

2. Σ_n-invariants of $H^{(n-1)(N-1)}(F(\mathbb{R}^N, n); \Bbbk)$, N even

The computation of $H^*(F(\mathbb{R}^N, n); \Bbbk)$ has been realized by F. Cohen in [6],

$$H^*(F(\mathbb{R}^N, n); \Bbbk) = \wedge_{1 \le i,j \le n} e_{i,j}/I$$

where I is the ideal generated by the elements $e_{i,j} - (-1)^N e_{j,i}$ and the elements $e_{i,j}e_{j,k} + e_{j,k}e_{k,i} + e_{k,i}e_{i,j}$. Here the elements $e_{i,j}$ have all degree $N-1$. A basis of the cohomology is given by the products $e_{i_1,j_1}e_{i_2,j_2} \ldots e_{i_r,j_r}$ where for each s, $i_s < j_s$ and $j_1 < j_2 < \ldots < j_r$. The permutation group Σ_n acts naturally on $\wedge(e_{i,j})$ by $\sigma(e_{i,j}) = e_{\sigma(i),\sigma(j)}$.

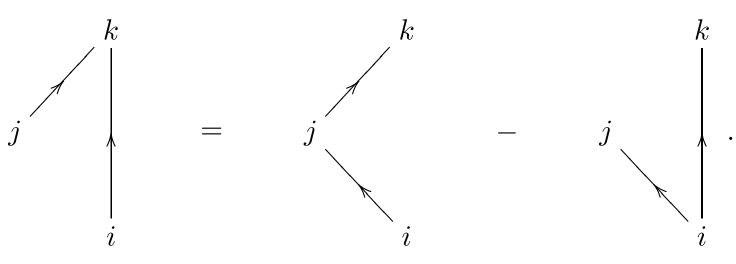
In this section, we prove

Theorem 5. *For $n \ge 3$, one has $H^{(n-1)(N-1)}(F(\mathbb{R}^N, n); \Bbbk)^{\Sigma_n} = 0$.*

In [17], E. Ossa makes a decomposition of the homology of the configuration space $F(R^N, n)$ as a module on the ring group $\mathbb{Z}[\Sigma_n]$, see also [1, Lemma 5.2]. As in [17], our starting point is similar: we replace the study of configuration spaces in the setting of trees.

We denote by $\mathcal{G}_n$ the set of *connected* trees with n vertices, denoted $1, 2, \ldots, n$, and with edges (i, j) naturally oriented from i to j if $i < j$. For sake of simplicity we will always denote the edges (i, j) with $i < j$.

Define G_n as the the quotient of the vector space over $\Bbbk$, constructed on the elements of $\mathcal{G}_n$, by the following relation: if a graph G contains the edges (i, k) and (j, k) with $i < j$, then $G = G' - G''$, where G' is obtained by replacing the above edges by the edges (i, j) and (j, k) and G'' is obtained by replacing the above edges by the edges (i, k) and (i, j), i.e.:



Observe that an iteration of this relation allows a choice of representing elements such that each vertex is the end of at most one edge.

The group Σ_n acts on $\mathcal{G}_n$ par permutation of vertices and this action induces an action on the vector space G_n. Since a tree is completely determined by the sequence

of its edges, we associate to a tree with edges (i_s, j_s), $s = 1, \ldots, n-1$, the element $e_{i_1,j_1} e_{i_2,j_2} \ldots e_{i_{n-1},j_{n-1}}$ of $\wedge(e_{i,j})$. This defines an isomorphism of Σ_n-vector spaces

$$G_n \xrightarrow{\cong} H^{(n-1)(N-1)}(F(\mathbb{R}^N, n); k)\,.$$

The proof of Theorem 5 is reduced to the study of $G_n^{\Sigma_n}$.

We now consider the stabilizator Σ_{n-1} of 1 in Σ_n and we denote by C_n the graph in $\mathcal{G}_n$ whose edges are $(1,2), (1,3), \ldots, (1, n-1)$.

Lemma 4. *For $n \geq 3$, one has $G_n^{\Sigma_{n-1}} = k\, C_n$.*

Proof. We suppose by induction on n that this is true for $k \leq n-1$. We then decompose G_n as a direct sum

$$G_n = \oplus_{p \leq n-1} G_{n,p}\,,$$

where $G_{n,p}$ is the sub-vector space generated by the trees T whose components T_1 and T_2 of 1 and 2 in $T \backslash (1,2)$ contain respectively $n-p$ and p vertices. The wedge $T_1 \vee T_2$ injects into T and we identify each T_i with its image.

Let $a \in G_n^{\Sigma_{n-1}}$, then $a = \sum_p a_p$, $a_p \in G_{n,p}$ and each a_p is invariant by the subgroup Σ_{n-2} that fixes the vertices 1 and 2. In particular, via the injection $T_1 \hookrightarrow T$, the group Σ_{n-p-1} acts on $G_{n,p}$, and by restriction we have a surjective map

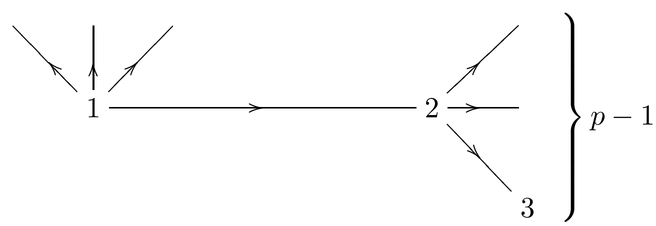
$$G_{n,p}^{\Sigma_{n-p-1}} \to G_{n-p}^{\Sigma_{n-p-1}}\,.$$

By induction this implies that a_p is a linear combination of trees in which $T_1 = C_{n-p}$. That means that the trees in this decomposition have the same form but not the same indexing on vertices. In the same way, we can suppose that $T_2 = C_p$. We consider now the index 3 and we can write

$$a = \sum_i \alpha_i a_i + \sum_j \beta_j a'_j$$

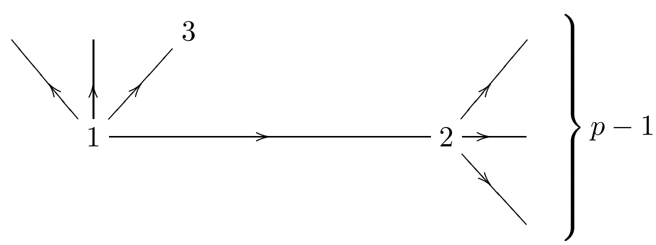
where the a_i are trees T with $T_1 = C_{n-p}$, $T_2 = C_p$ and $3 \in T_2$

a_i



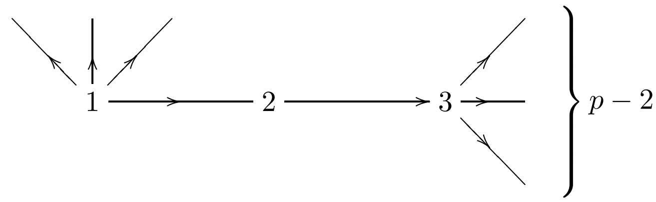
and the a'_j are trees T with $T_1 = C_{n-p}$, $T_2 = C_p$ and $3 \in T_1$.

a'_i



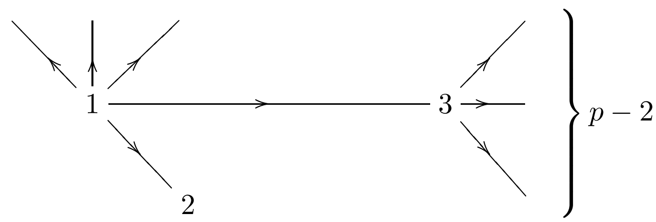
Denote by σ the permutation $(2,3)$ of the vertices 2 and 3. Then the image by σ of the graph a_i is the linear combination $a_{i,1} - a_{i,2}$ with

$a_{i,1}$



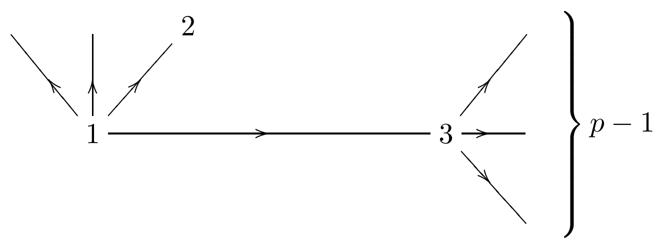
and

$a_{i,2}$

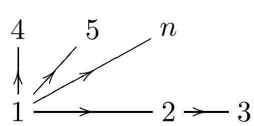


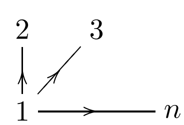
The image of a'_j by σ is a graph of the form

$\sigma(a'_j)$

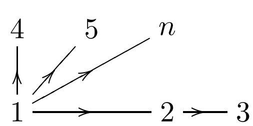


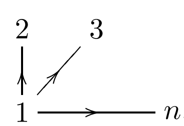
If we decompose each element $a_{i,1}$, $a_{i,2}$, $\sigma(a'_j)$ in the direct sum $\oplus G_{n,p}$ one has $a_{i,2} \in G_{n,1}$, $\sigma(a'_j) \in G_{n,1}$ and $a_{i,1} \in G_{n,p}$. Therefore, the equality $a = \sigma(a)$ implies $\alpha_i = 0$ for $a_i \in G_{n,p}$, $p \geq 3$. But now, by a decreasing induction on p, one notices $\beta_j = 0$ for $a'_j \in G_{n,p}$, $p \geq 2$. Thus,

$a \quad = \quad \alpha$ 

$+ \quad \beta$ 

and

$\sigma(a) \quad = \quad \alpha$ 

$+ \quad (\beta - \alpha)$ 

Therefore $\alpha = 0$ and $G_n^{\Sigma_{n-1}} = \Bbbk\, C_n$. ∎

Proof of Theorem 5. Let $a \in G_n^{\Sigma_n}$, then $a = \alpha C_n$. Denote $\tau = (1,2)$. Since $\tau(a) = a$, α must be equal to 0 . ∎

3. Σ_n-invariants of the Cohen-Taylor algebra E_1, N even.

Let M be a compact even dimensional manifold of cohomology algebra $H = H^*(M; \Bbbk)$. The symmetric group Σ_j acts on $H^{\otimes j}$ by permutation of factors and we denote by $\Gamma^j H \subset H^{\otimes j}$ the subspace of Σ_j-invariant elements.

Recall from Section 2, the existence of a correspondence between connected trees with n vertices and $H^{(n-1)(N-1)}(F(M,n); \Bbbk)$. This correspondence can be extended to the E_1 term of the Cohen-Taylor spectral sequence. Grants to the relation $(p_i^*(a) - p_j^*(a)) \otimes e_{ij}$, one has

$$E_1 = \oplus_{g\in\mathcal{F}_n} H^{\otimes l(G)} \cdot G,$$

where $\mathcal{F}_n$ denotes the set of *forests* (i.e. disjoint union of trees) with n vertices which are the end of at most one edge and where $l(G)$ denotes the number of components of G. We will make more precise this isomorphism in a particular case of forests we are interested in.

Let $0 \leq r \leq n/2$ be an integer and denote by $\mathcal{P}_r$ the set of r-uples of disjoint pairs $I = \{(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)\}$ with $i_k < j_k$ for $k \leq r$ and $1 \leq i_1 < i_2 < \ldots < i_r < j_r \leq n$. (Disjoint pairs means $i_u \neq j_v$, $j_u \neq j_v$ for all u and v.)

To any $I \in \mathcal{P}_r$, we associate the element $e_I = e_{i_1,j_1} \cdots e_{i_r,j_r}$ and we note $I_1 = \{i_1, \ldots, i_r\}$, $I' = \{i_1, \ldots, i_r, j_1, \ldots, j_r\}$. For any fixed $I \in \mathcal{P}_r$, we define a linear map

$$\gamma_I : T^r(s^{N-1}H) \to T^r(H) \cdot e_I$$

by $\gamma_I(s^{N-1}x_1 \otimes \cdots \otimes s^{N-1}x_r) = (-1)^{(N-1)(|x_1|+2|x_2|+\cdots+r|x_r|)} x_1 \otimes \cdots \otimes x_r \cdot e_I$. We define a Σ_r-action on the domain and the target of γ_I by permuting the factors in $T^r(H)$, $T^r(s^{N-1}H)$ and the pairs in I. The map γ_I, being clearly Σ_r-equivariant, induces a map

$$\gamma_I : \Gamma^r(s^{N-1}H) \to \Gamma^r H \cdot e_I\,.$$

We define $\phi_I : H^{\otimes n-2r} \to H^{\otimes n}$ by inserting the element $1 \in H^0$ in the position belonging to I' and the map $\psi_I : \Gamma^{\otimes r}(s^{N-1}H) \to H^{\otimes n} \cdot e_I$ by composition of γ_I with the insertion of the element 1 in position not in I_1.

The central object of this section is the map

$$\Phi_r : \Gamma^{n-2r} H \otimes \Gamma^r(s^{N-1}H) \to E_1$$

defined by

$$\Phi_r(\alpha \otimes \beta) = \sum_{I\in\mathcal{P}_r} \phi_I(\alpha) \cdot \psi_I(\beta)\,.$$

Theorem 6. *The image of the map $\Phi = \oplus_{r=0}^{[n/2]} \Phi_r$,*

$$\Phi : \oplus_{r=0}^{[n/2]} \left(\Gamma^{n-2r} H \otimes \Gamma^r(s^{N-1}H)\right) \to E_1$$

is the space $E_1^{\Sigma_n}$ of Σ_n-invariant elements of E_1.

Proof. Recall $E_1 \cong \oplus_{G\in\mathcal{F}_n} H^{\otimes l(G)} \cdot G$. where $l(G)$ denotes the number of components of G.

Let $\mathcal{S} = (s_1, s_2, \ldots, s_q)$ be a sequence of integers with $1 \leq s_1 \leq s_2 \leq \cdots \leq s_q$. We denote by $\Gamma_{\mathcal{S}}$ the set of elements in $\mathcal{F}_n$ with q components having respectively $s_1, s_2, \ldots,$ and s_q vertices. The direct sum $\oplus_{G\in\Gamma_{\mathcal{S}}} H^{\otimes q} \cdot G$ is a Σ_n-invariant subspace. If $a \in E_1^{\Sigma_n}$, then we may decompose a in $a = \sum_{\mathcal{S}} a_{\mathcal{S}}$, $a_{\mathcal{S}} \in (\oplus_{G\in\Gamma_{\mathcal{S}}} H^{\otimes q} \cdot G)^{\Sigma_n}$.

Let fix $\mathcal{S}$ and suppose $s_q > 1$. For any sequence $1 \leq n_1 < n_2 < \cdots < n_{s_q} \leq n$ we write $\Gamma_{\mathcal{S}} = \Gamma_1 \cup \Gamma_2$ where Γ_1 denotes the set of forest in $\Gamma_{\mathcal{S}}$ in which the vertices $n_1, \dots, n_{s_q}$ are vertices of a same tree. We consider the group Σ_{s_q} consisting of the permutations of the elements $n_1, \dots, n_{s_q}$. This is a subgroup of Σ_n and the subspaces $\oplus_{G\in\Gamma_1} H^{\otimes q} \cdot G$ and $\oplus_{G\in\Gamma_2} H^{\otimes q} \cdot G$ are invariant by the action of Σ_{s_q}. By Theorem 5 and its interpretation in terms of trees, one has $(\oplus_{G\in\Gamma_1} G)^{\Sigma_{s_q}} = 0$ if $s_q > 2$. Therefore, by arguing on each element of $H^{\otimes q}$, one deduces $(\oplus_{G\in\Gamma_1} H^{\otimes q} \cdot G)^{\Sigma_{s_q}} = 0$ if $s_q > 2$. Since this is true for any sequence $n_1 < n_2 \cdots < n_{s_q}$, we have

$$(\oplus_{G\in\Gamma_{\mathcal{S}}} H^{\otimes q} \cdot G)^{\Sigma_n} = 0 \text{ and } E_1^{\Sigma_n} \cong (\oplus_{G\in\mathcal{G}_1} H^{\otimes l(G)} \cdot G)^{\Sigma_n}\,,$$

where $\mathcal{G}_1$ denotes the set of forests G in which each component has at most two vertices.

Let $G \in \mathcal{G}_1$. Denote by $I = \{(i_1, j_1), (i_2, j_2), \dots, (i_r, j_r)\}$ the element of $\mathcal{P}_r$ built from the components with exactly two elements. We have clearly an isomorphism of Σ_n-vector spaces

$$\oplus_{G\in\mathcal{G}_1} H^{\otimes l(G)} \cdot G = \oplus_{r=0}^{[n/2]} \oplus_{I\in\mathcal{P}_r} H^{\otimes n-2r} \otimes H^{\otimes r} \cdot e_I\,,$$

where the components in $H^{\otimes n-2r}$ correspond to the components not in I', and the components in $H^{\otimes r}$ correspond to the components in I', ordered by $i_1 < i_2 < \cdots < i_r$. With the previous notation, we thus have

$$E_1^{\Sigma_n} = \oplus_{r=0}^{[n/2]} \left(\oplus_{I\in\mathcal{P}_r} H^{\otimes n-2r} \otimes H^r \cdot e_I\right)^{\Sigma_n}\,.$$

If we fix $I \in \mathcal{P}_r$, then each permutation σ of the set $\{1, \cdots, n\}\backslash I'$ and each permutation τ of the set of pairs in I preserve $H^{\otimes n-2r} \otimes H^r \cdot e_I$ and $\oplus_{J\neq I} H^{\otimes n-2r} \otimes H^{\otimes r} \cdot e_J$. Therefore,

$$\left(\oplus_{I\in\mathcal{P}_r} H^{\otimes n-2r} \otimes H^r \cdot e_I\right)^{\Sigma_n} \subset \oplus_{I\in\mathcal{P}_r} \Gamma^{n-2r} H \otimes \Gamma^r H \cdot e_I\,.$$

From the identification $(E_1, d_1) = \sum_{G\in\mathcal{F}_n} H^{\otimes l(G)} \cdot G$, we observe that if $\alpha \otimes \beta \cdot e_I \in \Gamma^{n-2r} H \otimes \Gamma^r H \cdot e_I$ and $\sigma \in \Sigma_n$, then $\sigma(\alpha \otimes \beta \cdot e_I) = \alpha \otimes \beta \cdot e_{\sigma(I)}$ and $\sigma(I) \in \mathcal{P}_r$. This implies that $\Phi : \oplus_{r=0}^{[n/2]} \left(\Gamma^{n-2r} H \otimes \Gamma^r(s^{N-1}H)\right) \to E_1^{\Sigma_n}$ is an isomorphism. $\square$

We define now a multiplication μ on $\oplus_{r=0}^{[n/2]} \left(\Gamma^{n-2r} H \otimes \Gamma^r(s^{N-1}H)\right)$ that makes Φ an isomorphism of algebras.

The graded vector space $\Gamma^* H = \oplus_{m\geq 0} \Gamma^m H$ is a sub-Hopf algebra of the Hopf algebra $T(H)$:

— the multiplication, $* : \Gamma^p H \otimes \Gamma^q H \to \Gamma^{p+q} H$, is the shuffle product;
— the comultiplication $\nabla : \Gamma^m H \to \oplus_{p+q=m} \Gamma^p H \otimes \Gamma^q H$ is defined by
$\nabla = \oplus_{p+q=m} \nabla_{p,q}$, $\qquad \nabla_{p,q}(x_1 x_2 \dots x_m) = x_1 \dots x_p \otimes x_{p+1} \dots x_m$.

By multiplying 2 by 2 the elements of $T^{2r}(H)$, we define a linear map $T^{2r}(H) \to T^r(H)$

$$x_1 \otimes \dots \otimes x_{2r} \mapsto (x_1 \bullet x_2) \otimes \dots \otimes (x_{2r-1} \bullet x_{2r})$$

that restricts to a map

$$red : \Gamma^{2r} H \to \Gamma^r H\,.$$

Finally, we quote two other structures:

— with the multiplication component by component, $\nu : T^n(H) \otimes T^n(H) \to T^n(H)$, the spaces $T^n(H)$ and $\Gamma^n H$ are graded commutative algebras.

— a natural action $\bar\nu$ of the algebra $\Gamma^n H$ on $\Gamma^n(s^{N-1}H)$ can be defined by
$(a_1\otimes\ldots\otimes a_n)\cdot(s^{N-1}b_1\otimes\ldots\otimes s^{N-1}b_n)=\varepsilon\ \ s^{N-1}(a_1\bullet b_1)\otimes\ldots\otimes s^{N-1}(a_n\bullet b_n)$,
with $\varepsilon=(-1)^{\sum_{j=2}^n|a_j|(|b_1|+\cdots+|b_{j-1}|)+\sum_{i=1}^n i|a_i|}$.

We can now state:

Theorem 7. *Define a multiplication μ on $\oplus_{r=0}^{[n/2]}\Gamma^{n-2r}H\otimes\Gamma^r(s^{N-1}H)$ by*

$$\mu:\left[\Gamma^{n-2r}H\otimes\Gamma^r(sH)\right]\otimes\left[\Gamma^{n-2s}H\otimes\Gamma^s(sH)\right]\to\Gamma^{n-2r-2s}H\otimes\Gamma^{r+s}(sH),$$

$$\mu((\alpha\otimes\beta)\otimes(\gamma\otimes\delta))=\sum_{ij}\varepsilon_{ij}\ \nu(\alpha_i\otimes\gamma_j)\otimes\bar\nu(red(\gamma'_j)\otimes\beta)*\bar\nu(red(\alpha'_i)\otimes\delta)\,,$$

where $s=s^{N-1}$, $\nabla_{n-2r-2s,2s}(\alpha)=\sum_i\alpha_i\otimes\alpha'_i$, $\nabla_{n-2r-2s,2r}(\gamma)=\sum_j\gamma_j\otimes\gamma'_j$ and ε_{ij} is the graded sign of the permutation. With this structure, Φ becomes an isomorphism of graded algebras.

This is a consequence of

Lemma 5. *The following diagram is commutative*

$$\begin{tikzcd}
{[\Gamma^{n-2r}H\otimes\Gamma^r(s^{N-1}H)]\otimes[\Gamma^{n-2s}H\otimes\Gamma^s(s^{N-1}H)]} \arrow[r, "\Phi_r\otimes\Phi_s"] \arrow[d, "\mu"'] & E_1\otimes E_1 \arrow[d, "mult"] \\
\Gamma^{n-2r-2s}H\otimes\Gamma^{r+s}(s^{N-1}H) \arrow[r, "\Phi_{r+s}"] & E_1
\end{tikzcd}$$

Proof. We denote by $\cdot$ the operations ν and $\bar\nu$. A simple computation gives

$$\begin{aligned}
&\Phi_r(\alpha\otimes\beta)\cdot\Phi_s(\gamma\otimes\delta)\\
&=mult\left(\textstyle\sum_{I\in\mathcal{P}_r}\phi_I(\alpha)\cdot\psi_J(\beta),\sum_{I\in\mathcal{P}_s}\phi_J(\gamma)\cdot\psi_J(\delta)\right)\\
&=\textstyle\sum_{ij}\sum_{K\in\mathcal{P}_{r+s}}\left(\sum_{I\cup J=K}\phi_K(\alpha_i)\cdot\phi_K(\gamma_j)\cdot\psi_I(\beta\cdot red(\gamma'_j))\cdot\psi_J(red(\alpha'_i)\cdot\delta)\right)\\
&=\textstyle\sum_{ij}\sum_{K\in\mathcal{P}_{r+s}}\phi_K(\alpha_i\gamma_j)\cdot\psi_K((\beta\cdot red(\gamma'_j)*(red(\alpha'_i)\cdot\delta))\\
&=\Phi_{r+s}(\mu((\alpha\otimes\beta)\otimes(\gamma\otimes\delta))).
\end{aligned}$$

□

Remark 6. The cardinality of the set $\mathcal{P}_r$ is $|\mathcal{P}_r|=\dfrac{n!}{(n-2r)!2^r r!}$.

For justifying this formula, denote by Q_r the set of ordered r-uples of disjoint pairs. We have clearly $|Q_r|=r!|\mathcal{P}_r|$. If we fix an element in Q_r, we obtain an element of Q_{r+1} by choosing 2 elements between the remaining $n-2r$ variables. Therefore,

$$|Q_{r+1}|=\frac{(n-2r)(n-2r-1)}{2}|Q_r|\,,$$

and thus,

$$|\mathcal{P}_{r+1}|=\frac{(n-2r)(n-2r-1)}{2(r+1)}|\mathcal{P}_r|\,.$$

An induction on r based on the last formula gives the result.

4. The differential graded algebra C_n^H, N even

The purpose of this section is to use the existence of an isomorphism between the subspace of invariants $\Gamma^n H$ and the exterior algebra $\wedge^n H$ for having a better description of the differential graded algebra $(E_1, d_1)^{\Sigma_n}$. This will impose some restriction on the field $\Bbbk$. *For all this section we suppose that $\Bbbk = \mathbb{Q}$ or $\Bbbk = \mathbb{F}_p$ with $p > n$.*

Let $\rho_n : \wedge^n H \to \Gamma^n H$ be the symmetrization map defined by

$$\rho_n(x_1 \wedge x_2 \wedge \cdots \wedge x_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \varepsilon_\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

where ε_σ is the graded sign of the permutation

$$x_1 \dots x_n \mapsto x_{\sigma(1)} \dots x_{\sigma(n)} .$$

The map ρ_n is an isomorphism which sends the usual multiplication $m_\wedge$ of $\wedge^n H$ on the shuffle product on $\Gamma^n H$. We have now to translate on

$$C_n^H = \oplus_{r=0}^{[n/2]} \wedge^{n-2r} H \otimes \wedge^r (s^{N-1} H)$$

the different structures we used for constructing the law of $\oplus_{r=0}^{[n/2]} \Gamma^{n-2r} H \otimes \Gamma^r (s^{N-1} H)$.

Theorem 8. The multiplication law

$$\left[\wedge^{n-2r} H \otimes \wedge^r(sH)\right] \otimes \left[\wedge^{n-2s} H \otimes \wedge^s(sH)\right] \to \wedge^{n-2r-2s} H \otimes \wedge^{r+s}(sH)$$

is defined by

$$(x_1 \wedge \ldots \wedge x_{n-2r} \otimes sy_1 \wedge \ldots \wedge sy_r) \cdot (z_1 \wedge \ldots \wedge z_{n-2s} \otimes st_1 \wedge \ldots \wedge st_s) = \frac{1}{(n-2s-2r)!} \sum_{\substack{\sigma \in \Sigma_{n-2r} \\ \tau \in \Sigma_{n-2s}}} \varepsilon_{\sigma,\tau}\ \alpha_{\sigma,\tau} \otimes \beta_{\sigma,\tau} \wedge \gamma_{\sigma,\tau},$$

where:

— $s = s^{N-1}$ *and* $\varepsilon_{\sigma,\tau}$ *denotes the graded sign of the permutation,*
— $\alpha_{\sigma,\tau}$, $\beta_{\sigma,\tau}$ *and* $\gamma_{\sigma,\tau}$ *are defined by:*
$\alpha_{\sigma,\tau} = (x_{\sigma(1)} \bullet z_{\tau(1)}) \wedge \ldots \wedge (x_{\sigma(n-2r-2s)} \bullet z_{\tau(n-2r-2s)})$,
$\beta_{\sigma,\tau} = s(x_{\sigma(n-2r-2s+1)} \bullet x_{\sigma(n-2r-2s+2)} \bullet t_1) \wedge \ldots \wedge s(x_{\sigma(n-2r-1)} \bullet x_{\sigma(n-2r)} \bullet t_s)$,
$\gamma_{\sigma,\tau} = s(z_{\tau(n-2r-2s+1)} \bullet z_{\tau(n-2r-2s+2)} \bullet y_1) \wedge \ldots \wedge s(z_{\tau(n-2s-1)} \bullet z_{\tau(n-2s)} \bullet y_r)$.

We first define *a multiplication* $\nu' : \wedge^n H \otimes \wedge^n H \to \wedge^n H$ *by*

$$\nu'(x_1 \wedge \cdots \wedge x_n \otimes y_1 \wedge \cdots \wedge y_n) = \sum_{\sigma \in \Sigma_n} \varepsilon_\sigma \ (x_1 \bullet y_{\sigma(1)}) \wedge \cdots \wedge (x_n \bullet y_{\sigma(n)}) ,$$

which makes commutative the following diagram

$$\begin{tikzcd}
\wedge^n H \otimes \wedge^n H \arrow[r, "\rho_n \otimes \rho_n"] \arrow[d, "\nu'"'] & \Gamma^n H \otimes \Gamma^n H \arrow[d, "\nu"] \\
\wedge^n H \arrow[r, "\rho_n"'] & \Gamma^n H
\end{tikzcd}$$

This *multiplication extends into an action, denoted* $\bar\nu'$ *of* $\wedge^n H$ *on* $\wedge^n(s^{N-1}H)$. This action corresponds, via ρ to the action $\bar\nu$ of $\Gamma^n H$ on $\Gamma^n(s^{N-1}H)$, i.e. the following

diagram trivially commutes

$$
\begin{tikzcd}
\wedge^n H\otimes \wedge^n(s^{N-1}H) \arrow[r, "\rho_n\otimes\rho_n"] \arrow[d, "\bar{\nu}'"'] & \Gamma^n H\otimes\Gamma^n(s^{N-1}H) \arrow[d, "\bar{\nu}"] \\
\wedge^n(s^{N-1}H) \arrow[r, "\rho_n"'] & \Gamma^n(s^{N-1}H)
\end{tikzcd}
$$

A classical diagonal map $\Delta'_{p,q}:\wedge^{p+q}H\to\wedge^p H\otimes\wedge^q H$ is defined by

$$\Delta'_{p,q}(x_1\wedge\cdots\wedge x_{p+q})=\sum_{\sigma\in(p,q)Sh}\varepsilon_\sigma\left(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(p)}\right)\otimes\left(x_{\sigma(p+1)}\wedge\cdots\wedge x_{\sigma(p+q)}\right),$$

where $(p,q)Sh$ denotes the set of (p,q) shuffles of the set $\{1,2,\cdots,p+q\}$. Since we work in free commutative graded algebras, we have also

$$\Delta'_{p,q}(x_1\wedge\cdots\wedge x_{p+q})=\frac{1}{p!q!}\sum_{\sigma\in\Sigma_{p+q}}\varepsilon_\sigma(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(p)})\otimes\left(x_{\sigma(p+1)}\wedge\cdots\wedge x_{\sigma(p+q)}\right).$$

The symmetrization map ρ gives a relation between $\Delta'_{p,q}$ and $\Delta_{p,q}$ in terms of a commutative diagram

$$
\begin{tikzcd}
\wedge^{p+q}H \arrow[r, "\rho"] \arrow[d, "\Delta'_{p,q}"'] & \Gamma^{p+q}H \arrow[d, "\Delta_{p,q}"] \\
\wedge^p H\otimes\wedge^q H \arrow[r, "\rho\otimes\rho"'] & \Gamma^p H\otimes\Gamma^q H
\end{tikzcd}
$$

Finally *a reduction map*

$$red':\wedge^{2r}H\to\wedge^r H$$

is defined by

$$
\begin{aligned}
red'(x_1\wedge\cdots\wedge x_{2r}) &= 2^r\sum_{I\in\mathcal{P}_r}\varepsilon_I\,(x_{i_1}\bullet x_{j_1})\wedge\cdots\wedge(x_{i_r}\bullet x_{j_r})\\
&= \frac{1}{r!}\sum_{\sigma\in\Sigma_n}\varepsilon_\sigma\left(x_{\sigma(1)}\bullet x_{\sigma(2)}\right)\wedge\cdots\wedge\left(x_{\sigma(2r-1)}\bullet x_{\sigma(2r)}\right).
\end{aligned}
$$

This reduction map makes the following diagram commutative

$$
\begin{tikzcd}
\wedge^{2r}H \arrow[r, "\rho_{2r}"] \arrow[d, "red'"'] & \Gamma^{2r}H \arrow[d, "red"] \\
\wedge^r H \arrow[r, "\rho_r"'] & \Gamma^r H
\end{tikzcd}
$$

Proof of Theorem 8. The operations ν', Δ' and red' fit together into the following commutative diagram:

$$
\begin{tikzcd}
\wedge^{2n-r}H\otimes\wedge^r(sH)\otimes\wedge^{n-2s}H\otimes\wedge^s(sH) \arrow[r, "\rho^{\otimes 4}"] \arrow[d, "q'"'] & \Gamma^{n-2r}H\otimes\Gamma^r(sH)\otimes\Gamma^{n-2s}H\otimes\Gamma^s(sH) \arrow[d, "q"] \\
\wedge^{n-2r-2s}H\otimes\wedge^r(sH)\otimes\wedge^s(sH) \arrow[r, "\rho^{\otimes 3}"] \arrow[d, "1\otimes m_\wedge"'] & \Gamma^{n-2r-2s}H\otimes\Gamma^r(sH)\otimes\Gamma^s(sH) \arrow[d, "1\otimes *"] \\
\wedge^{n-2r-2s}H\otimes\wedge^{r+s}(sH) \arrow[r, "\rho^{\otimes 2}"'] & \Gamma^{n-2r-2s}H\otimes\Gamma^{r+s}(sH)
\end{tikzcd}
$$

where $sH = s^{N-1}H$, m is the usual multiplication in $\wedge(-)$, $*$ is the shuffle product, and q and q' are defined by

$$q = (\nu\otimes\bar\nu\otimes\bar\nu)\circ T\circ(1\otimes red\otimes 1\otimes 1\otimes red\otimes 1)\circ(\Delta_{n-2r-2s,2s}\otimes 1\otimes\Delta_{n-2r-2s,2r}\otimes 1)$$

$$q' = (\nu'\otimes\bar\nu'\otimes\bar\nu')\circ T'\circ(1\otimes red'\otimes 1\otimes 1\otimes red'\otimes 1)\circ(\Delta'_{n-2r-2s,2s}\otimes 1\otimes\Delta'_{n-2r-2s,2r}\otimes 1)$$

The morphisms T and T' are the permutation maps

$$T:\Gamma^lH\otimes\Gamma^sH\otimes\Gamma^r sH\otimes\Gamma^lH\otimes\Gamma^rH\otimes\Gamma^s sH\to(\Gamma^lH)^{\otimes 2}\otimes\Gamma^rH\otimes\Gamma^r sH\otimes\Gamma^sH\otimes\Gamma^s sH$$

$$T':\wedge^lH\otimes\wedge^sH\otimes\wedge^r sH\otimes\wedge^lH\otimes\wedge^rH\otimes\wedge^s sH\to(\wedge^lH)^{\otimes 2}\otimes\wedge^rH\otimes\wedge^r sH\otimes\wedge^sH\otimes\wedge^s sH\,,$$

with $l = n-2r-2s$.

The composition on the right is the multiplication μ on $\oplus_{r=0}^{[n/2]}\Gamma^{n-2r}H\otimes\Gamma^r(s^{N-1}H)$, therefore the composition on the left is a multiplication μ' on C_n^H making ρ a morphism of algebras. To determine explicitely μ', let

$$x_1\wedge\cdots\wedge x_{n-2r}\otimes sy_1\wedge\cdots\wedge sy_r\otimes z_1\wedge\cdots\wedge z_{n-2s}\otimes st_1\wedge\cdots\wedge st_s$$

be an element of $\wedge^{2n-r}H\otimes\wedge^r(sH)\otimes\wedge^{n-2s}H\otimes\wedge^s(sH)$. By applying $\Delta'_{n-2r-2s,2s}\otimes 1\otimes\Delta'_{n-2r-2s,2r}\otimes 1$, we obtain

$$\frac{1}{(n-2r-2s)!(2s)!(n-2r-2s)!(2r)!}\sum_{\sigma\in\Sigma_{n-2r},\tau\in\Sigma_{n-2s}}$$
$$\begin{aligned}&x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(n-2r-2s)}\otimes x_{\sigma(n-2r-2s+1)}\wedge\cdots\wedge x_{\sigma(n-2r)}\otimes sy_1\wedge\cdots\wedge sy_r\\ &\otimes z_{\tau(1)}\wedge\cdots\wedge z_{\tau(n-2r-2s)}\otimes z_{\tau(n-2r-2s+1)}\wedge\cdots\wedge z_{\tau(n-2s)}\otimes st_1\wedge\cdots\wedge st_s\end{aligned}$$

We apply now the reduction process $1\otimes red'\otimes 1\otimes 1\otimes red'\otimes 1$ and we obtain

$$\frac{1}{(n-2r-2s)!(s)!(n-2r-2s)!(r)!}\sum_{\sigma\in\Sigma_{n-2r},\tau\in\Sigma_{n-2s}}$$
$$\begin{aligned}&x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(n-2r-2s)}\\ &\otimes(x_{\sigma(n-2r-2s+1)}\bullet x_{\sigma(n-2r-2s+2)})\wedge\cdots\wedge(x_{\sigma(n-2r-1)}\bullet x_{\sigma(n-2r)})\\ &\otimes sy_1\wedge\cdots\wedge sy_r\otimes z_{\tau(1)}\wedge\cdots\wedge z_{\tau(n-2r-2s)}\\ &\otimes(z_{\tau(n-2r-2s+1)}\bullet z_{\tau(n-2r-2s+2)})\wedge\cdots\wedge(z_{\tau(n-2s-1)}\bullet z_{\tau(n-2s)})\otimes st_1\wedge\cdots\wedge st_s.\end{aligned}$$

The last step is the composition with $(\nu'\otimes\bar\nu'\otimes\bar\nu')\circ T'$ which gives

$$\frac{1}{(n-2r-2s)!}\sum_{\sigma\in\Sigma_{n-2r},\tau\in\Sigma_{n-2s}}\varepsilon_{\sigma,\tau}$$
$$(x_{\sigma(1)}\bullet z_{\tau(1)})\wedge\cdots\wedge(x_{\sigma(n-2r-2s)}\bullet z_{\tau(n-2r-2s)})$$
$$\otimes s(y_1\bullet z_{\tau(n-2r-2s+1)}\bullet z_{\tau(n-2r-2s+2)})\wedge\cdots\wedge s(y_r\bullet z_{\tau(n-2s-1)}\bullet z_{\tau(n-2s)})$$
$$\otimes s(x_{\sigma(n-2r-2s+1)}\bullet x_{\sigma(n-2r-2s+2)}\bullet t_1)\wedge\cdots\wedge s(x_{\sigma(n-2r-1)}\bullet x_{\sigma(n-2r)}\bullet t_s).$$

This proves the formula for the product. □

Before doing the computation of the differential, we need to determine a system of generators of this algebra:

Proposition 7. *Denote by e_0 the unit of H and by $\cdot$ the multiplication in C_n^H. Then $\frac{e_0}{n!}$ is the unit of C_n^H and*

$$\left(\frac{e_0^{n-2r}}{(n-2r)!}\otimes sy_1\wedge\cdots\wedge sy_r\right)\cdot\left(\frac{e_0^{n-2}}{(n-2)!}\otimes sa\right)=\frac{e_0^{n-2r-2}}{(n-2r-2)!}\otimes sy_1\wedge\cdots\wedge sy_r\wedge sa\,.$$

Moreover, the algebra is generated by the elements of $\wedge^nH$ and by the elements $e_0^{n-2}\otimes sa$.

Proof. The two first properties are simple computations from the definition of the mutiplication. The last one follows from them and from the following relation

$$\left(\frac{e_0^{n-2r}}{(n-2r)!}\otimes sy_1\wedge\cdots\wedge sy_r\right)\cdot\left(\frac{e_0^{2r}}{(2r)!}\wedge x_1\wedge\cdots\wedge x_{n-2r}\right)-$$
$$(x_1\wedge\cdots\wedge x_{n-2r}\otimes sy_1\wedge\cdots\wedge sy_r)\in\wedge H\otimes(\wedge^rsH)^{>l}\ ,$$

where the elements x_i are of degree >0 and l is the degree of $sy_1\wedge\cdots\wedge sy_r$. A decreasing induction on the degree of elements in $\wedge^rsH$ ends the proof. □

We may also observe directly from Theorem 8 that the multiplication law induced on $\wedge^nH$ is given by

$$(a_1\wedge\cdots\wedge a_n)\cdot(b_1\wedge\cdots\wedge b_n)=\sum_{\sigma\in\Sigma_n}\varepsilon_\sigma\ (a_{\sigma(1)}\bullet b_1)\wedge\cdots\wedge(a_{\sigma(n)}\bullet b_n).$$

We now arrive to the *description of the differential.* Recall first that $H=\oplus_{p=0}^NH^p$, with $H^N=\Bbbk\Omega$. The diagonal class Δ is the element of $H\otimes H$ defined by

$$\Delta=\sum_i(-1)^{|b_i'|}b_i\otimes b_i'$$

where the $\{b_i\}$ form an homogeneous basis for H and the $\{b_i'\}$ form the dual basis defined by

$$b_i\bullet b_j'=\delta_{ij}\Omega\,.$$

By Poincaré duality, for each element $h\in H$ we have $(h\otimes 1)\cdot\Delta=(1\otimes h)\cdot\Delta$.

Definition 8. We define a differential d on $\oplus_{r=0}^{[n/2]}\wedge^{n-2r}H\otimes\wedge^r(s^{N-1}H)$ by

$$d(x_1\wedge\cdots\wedge x_{n-2r}\otimes sy_1\wedge\cdots\wedge sy_r)=$$

$$\frac{1}{2}\sum_{i=1}^r\sum_k(-1)^{|b_k'|+i+\sum_{j=1}^{i-1}|y_j|}x_1\wedge\cdots\wedge x_{n-2r}\wedge(y_i\bullet b_k)\wedge b_k'\otimes sy_1\wedge\cdots\wedge\widehat{sy_i}\wedge\cdots\wedge sy_r\,.$$

Proof of Theorem 1. We are reduced to prove the commutativity of the following diagram

$$\begin{array}{ccc} \wedge^{n-2r}H \otimes \wedge^r(sH) & \xrightarrow{\Phi} & E_1 \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d_1} \\ \wedge^{n-2r+2}H \otimes \wedge^{r-1}(sH) & \xrightarrow{\Phi} & E_1 \end{array}$$

From Proposition 7, it is enough to check it in the particular case $r = 1$ and $x_1 \wedge \cdots \wedge x_{n-2r} = e_0^{n-2}$. On one hand, we have:

$$d_1\Phi(e_0^{n-2} \otimes sa) = (n-2)! \sum_{i<j} \sum_k (-1)^{|b'_k|} 1 \otimes \cdots \otimes (a \bullet b_k) \otimes \cdots \otimes b'_k \otimes \cdots \otimes 1\,,$$

where the terms $(a \bullet b_k)$ and b'_k are respectively in positions i and j. On the other hand, we have:

$$\begin{aligned} \Phi d(e_0^{n-2} \otimes sa) = \;& \frac{(n-2)!}{2} \sum_{i<j} \sum_k (-1)^{|b'_k|}\; 1 \otimes \cdots \otimes (a \bullet b_k) \otimes \cdots \otimes b'_k \otimes \cdots \otimes 1 \\ & + \textstyle\sum_{i<j} \sum_k (-1)^{|b'_k| + |b'_k|\cdot|ab_k|}\; 1 \otimes \cdots \otimes b'_k \otimes \cdots \otimes (a \bullet b_k) \otimes \cdots \otimes 1\,. \end{aligned}$$

In the first sum the elements $(a \bullet b_k)$ and b'_k are in positions i and j, and in the second sum the elements b'_k and $(a \bullet b_k)$ are in positions i and j.

Since N is even and $\Delta = \sum_k (-1)^{|b'_k|} b_k \otimes b'_k = \sum_k (-1)^{|b_k| + |b'_k| + |b_k|} b'_k \otimes b_k$, we have

$$\sum_k (-1)^{|b'_k| + |ab_k|\cdot|b'_k|} b'_k \otimes (a \bullet b_k) = \sum_k (-1)^{N + |b'_k|} (a \bullet b_k) \otimes b'_k = \sum_k (-1)^{|b'_k|} (a \bullet b_k) \otimes b'_k\,.$$

Therefore $\Phi \circ d = d_1 \circ \Phi$. □

Proof of Theorem 3, N even. Observe that the underlying complex of the cdga (C_n^H, d) we have described before, is isomorphic to the complex defined by Y. Félix and J.-C. Thomas [11] for the determination of the homology vector space $H_*(C_n(M); \mathbb{Q})$. □

5. The space of unordered configurations in $P^2(\mathbb{C})$

In this section, we apply the previous results to the unordered configuration space of $P^2(\mathbb{C})$ and prove Theorem 2.

Let $H = H^*(P^2(\mathbb{C}); \mathbb{Q})$. As quoted in the introduction (see also Definition 8), the direct sum $(\oplus_n C_n^H, d)$ is, as a complex, the commutative differential graded algebra $(\wedge(x_0, x_1, x_2, y_0, y_1, y_2), d)$ with $|x_0| = 0$, $|x_1| = 2$, $|x_2| = 4$, $|y_0| = 3$, $|y_1| = 5$, $|y_2| = 7$ and the differential d defined by

$$d(x_0) = d(x_1) = d(x_2) = 0\,;\; d(y_0) = x_0x_2 + \frac{1}{2}x_1^2\,;\; d(y_1) = x_1x_2\,;\; d(y_2) = \frac{1}{2}x_2^2\,.$$

Lemma 9. *A basis of the reduced cohomology of $(\wedge(x_0, x_1, x_2, y_0, y_1, y_2), d)$ is given by the classes of the cocycles*

1. x_0^n, $x_0^{n-1}x_1$ *and* $x_0^{n-1}x_2$, *for* $n \geq 1$,
2. $x_0^{n-3}x_1y_1 - 2x_0^{n-3}x_2y_0 + 4x_0^{n-2}y_2$ *and* $x_0^{n-3}x_2y_1 - 2x_0^{n-3}x_1y_2$ *for* $n \geq 3$,
3. $x_0^{n-4}x_1x_2y_1 - 2x_0^{n-4}x_1^2y_2$ *for* $n \geq 4$.

Proof. We consider the relative minimal model (see [10])

$$(\wedge(x_0,0) \longrightarrow (\wedge(x_0,x_1,x_2,y_0,y_1,y_2),d) \xrightarrow{p} (\wedge(x_1,x_2,y_0,y_1,y_2),\bar d)\,.$$

Since x_1^2 and x_2^2 form a regular sequence in $\wedge(x_1,x_2)$, the canonical projection

$$(\wedge(x_1,x_2,y_0,y_1,y_2),\bar d) \to (\wedge(x_1,x_2,y_1)/(x_1^2,x_2^2),\bar d(y_1)=x_1x_2)$$

is a quasi-isomorphism. The algebra $\wedge(x_1,x_2,y_1)/(x_1^2,x_2^2)$ is finite dimensional and a basis of the reduced cohomology is given by the classes of the cocyles x_1, x_2, y_1x_1, y_1x_2, $y_1x_1x_2$. The morphism $H^*(p)$ is surjective because $H^*(p)([x_1y_1-2x_2y_0+4x_0y_2])=[x_1y_1]$, $H^*(p)([x_2y_1-2x_1y_2])=[x_1y_2]$ and $H^*(p)([x_1x_2y_1-2x_1^2y_2])=[y_1x_1x_2]$. Therefore the spectral sequence obtained by filtering the complex $(\wedge(x_0,x_1,x_2,y_0,y_1,y_2),d)$ by the powers of the ideal generated by x_0 collapses at the E_2-term. This gives the above basis for the cohomology. $\square$

Proof of Theorem 2. Let fix n. Then a linear basis of the cohomology of C_n^H is given by the classes

$$\begin{cases} x_0^n\,,\ x_0^{n-1}x_1\,,\ x_0^{n-1}x_2\,, \quad (n\geq 1)\,,\\ z_0=x_0^{n-3}x_1y_1-2x_0^{n-3}x_2y_0+4x_0^{n-2}y_2\,, \quad (n\geq 3)\,,\\ z_1=x_0^{n-3}x_2y_1-2x_0^{n-3}x_1y_2\,, \quad (n\geq 3)\,,\\ z_2=x_0^{n-4}x_1x_2y_1-2x_0^{n-4}x_1^2y_2\,, \quad (n\geq 4)\,. \end{cases}$$

Denote $t_0=\dfrac{x_0^n}{n!}$, $t_1=\dfrac{x_0^{n-1}x_1}{(n-1)!}$ and $t_2=\dfrac{x_0^{n-1}x_2}{(n-1)!}$. Then t_0 is the unit, and $t_1^2=(3-2n)t_2$. Moreover,

$$[t_1]\cdot[z_1]=(n-3)[z_2]$$

for $n\geq 4$ and

$$[t_1]\cdot[z_0]=(3-2n)[z_1]$$

for $n\geq 3$. This gives the statement. $\square$

6. Unordered configuration space of odd dimensional manifolds

Proof of Theorem 4. The manifold M being odd dimensional, one has $e_{ij}=-e_{ji}$ and the elements e_{ij} are of even degree. We have also to take in account the relation $e_{ij}^2=0$ which is automatic for degree reason when M is even dimensional.

Let $\alpha\in E_1^{\Sigma_n}=(H^{\otimes n}\otimes\wedge e_{ij}/I)^{\Sigma_n}$. We decompose α in $\alpha=\alpha_{n-1}+\cdots\alpha_0$ where $\alpha_r\in K_r=H^{\otimes n}\otimes e_{rn}\otimes\wedge(e_{ij})/I$ with the remaining e_{ij} such that $(i,j)\notin\{(r,n),(r+1,n)\cdots,(n-1,n)\}$. For instance, $\alpha_0\in H^{\otimes n}\otimes\wedge(e_{ij})_{i<j<n}/I$.

Let s be the greatest integer such that $\alpha_s\neq 0$. The image $\tau(\alpha)$ of α by the permutation (s,n) satisfies $\tau(\alpha)=-\alpha_s+\beta$ with $\beta\in\oplus_{i\neq s}K_i$. Therefore, if $\alpha\in E_1^{\Sigma_n}$, we have $\alpha_s=0$ and, by induction, $\alpha\in(H^{\otimes n})^{\Sigma_n}=\Gamma^nH$.

The first term of the spectral sequence $\mathcal{E}(M,n,\Bbbk)^{\Sigma_n}$ is *concentrated in filtration degree 0*. Therefore, the spectral sequence collapses and Γ^nH (which coincides with $\wedge^nH$ with our hypothesis on the field $\Bbbk$) is the algebra of cohomology $H^*(C_n(M);\Bbbk)$. $\square$

As a graded vector space this corresponds to the result previously obtained by C.F. Bödigheimer, F. Cohen and L. Taylor ([4]). Here we prove that, in the case of an odd dimensional manifold M, the cohomology *algebra* of unordered configurations depends only on Betti number of the manifold M.

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Département de Mathematiques, Université Catholique de Louvain, 2, Chemin du Cyclotron, 1348 Louvain-La-Neuve, Belgium

E-mail address: felix@uranus.sc.ucl.ac.be

Département de Mathematiques, UMR 8524, Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France

E-mail address: Daniel.Tanre@agat.univ-lille1.fr